DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

CONTINUOUS SOCIAL DECISION PROCEDURES

John A. Ferejohn
California Institute of Technology

and

Edward W. Packel
California Institute of Technology
and Lake Forest College

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ABSTRACT

Classical social decision procedures are supposed to map lists of preference orderings into binary relations which describe society's "preferences." But when there are infinitely many alternatives the resulting plethora of possible preference orderings make it impossible to differentiate "nearby" preference relations. If the preference information used to make social decisons is imperfect, society may wish to implement a continuous social decision procedure (SDP) so that nearby preference configurations will map into nearby social preference relations. It is shown here that a continuity requirement can severely restrict the admissable behavior of a social decision procedure. Furthermore a characterization of continuous SDP's is presented which facilitates the examination of such procedures and their relation to various voting mechanisms.

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I. INTRODUCTION

Classical social decision procedures are supposed to map lists of preference orderings into binary relations which describe society's "preferences." But when there are infinitely many alternatives the resulting plethora of possible preference orderings make it impossible to differentiate "nearby" preference relations. If, for whatever reason, the preference information used to make social decisions is imperfect, society may wish to implement a continuous social decision procedure (SDP) so that nearby preference configurations will map into nearby social preference relations. But, as will be shown, a continuity requirement can severely restrict the admissable behavior of a social decision procedure. A great many otherwise attractive decision procedures -- including virtually all of the rules which base social decision on the proportion of those individuals having a preference -- turn out not to satisfy this axiom for fairly natural topologies. In view of its strength, then, it seems sensible to ask in what sense a continuity property is a desirable characteristic of a social decison procedure.

There seem to be two principal justifications for a continuity requirement. First, people may not be able, psychologically or physiologically, to tell the difference between a given preference and one that is very close to it. In the event that this failure of discrimination is based on physiological limitations, one could claim that either of two nearby preferences ever has a claim to being considered the "true" preference. Thus, even if the social decision based on two nearby preferences were greatly different, it could be argued that there is no reason to believe that one has any greater claim to validity than the other. The social decision process is basically indeterminate in this case (indeed, it might best be modelled as a correspondence rather than a function) and, at points of discontinuity, the degree of indeterminacy may be relatively great.

On the other hand if the reason that an individual cannot tell the difference between two nearby preference relations is because the act of working out the necessary comparisons is costly, then there is a sense in which the individual's true preference "really" is distinct from nearby ones. And if the social decision process was discontinuous at such a point, the social decision could be quite "incorrect" from that standpoint. In this case continuity of the SDP would allow society to get as near to the correct social binary relation as is desired by paying the cost of making discriminations among preferences at the individual level.

The second argument in favor of the continuity of a social decision procedure is that if the transmission process by which individuals communicate their preference orderings is relatively coarse then two distinct preference configurations which might yield very different social decisions could produce the same messages from the individuals. This coarseness may be inherent in the nature of linguistic description itself but it seems likely that increasingly fine discriminations would be available at some cost. (For example, through the use of longer and more complete descriptions of preferences.)

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If the processes of discrimination or transmission are costly, then, but these costs are not infinite, the continuous social decision procedures have a natural appeal. These processes are the ones that can be guaranteed to be as nearly "correct" as we like if the appropriate costs are incurred to improve the discrimination or transmission processes. If, on the other hand, either of these processes have limits beyond which they cannot be improved then continuity of the SDP will not guarantee its approximately accurate performance. Indeed, the very notion of "accurate" performance may make little sense if such natural limits exist.

In this paper we shall present a characterization of the continuous SDP's that will reveal the crucial role played by the way in which an SDP treats individual indifference. Some impossibility theorems will then follow as corollaries to this result. We then show that if an SDP is continuous and satisfies binary independence and a weak monotonicity property, it must be a simple game at each point. Finally, we show that if an SDP is continuous-valued as well as continuous, then some of our theorems can be extended from pointwise to local results.

II. DEFINITIONS AND NOTATION

Throughout the paper we let X stand for the set of alternatives and N for the set of individuals. Individual i's preferences are described by a weak order over X and are denoted by R_i. As is usual, the asymmetric and symmetric parts of R_i are written as P_i and I_i. The collection of weak orders on X is written as $\mathcal{R}$,

while $\mathcal{B}$ denotes the collection of complete, reflexive binary relations on X.

Let $|N| = n$ and let each $i \in N$ have preferences in $\Omega_i \subseteq \mathcal{R}$. We assume throughout the paper that each Ω_i is rich enough that for any x, y $\in$ X, there exists R_i, R_i', and $R_i'' \in \Omega_i$ such that xP_iy, $xI_i'y$, and $yP_i''x$. Letting $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$, an SDP is a mapping $F : \Omega \to \mathcal{B}$. Given $\pi = (R_1, R_2, \ldots, R_n) \in \Omega$, we abbreviate $F(\pi)$ by B. Given $B \in \mathcal{B}$ and x, y $\in$ X, we use the shorthand $xPy \Leftrightarrow xBy$ and $\sim yBx$. Likewise, $xIy \Leftrightarrow xBy$ and yBx. An SDP is said to satisfy <u>binary independence</u> (BI) at (x,y) if $\forall \pi, \pi' \in \Omega$,

$$(xR_iy \Leftrightarrow xR_i'y \text{ and } yR_ix \Leftrightarrow yR_i'x \;\forall i \in N) \Rightarrow (xBy \Leftrightarrow xB'y).$$

We assume that $\mathcal{B}$ is endowed with a topology τ. Then each Ω_i will be given the relative topology and Ω the product topology, inherited from τ_i. The following topological conditions on $<\mathcal{B},\tau>$ will be important in what follows.

<u>T1</u>: $\forall x \neq y$, V a neighborhood of B and $xIy \Rightarrow \exists B' \in V$ s.t. $xP'y$.

<u>T2</u>: $\forall x, y \in X$, $xPy \Rightarrow \exists$ a neighborhood V of B s.t. $xP'y \;\; \forall B' \in V$.

We note that for natural definitions of Ω both of these conditions are satisfied by preference topologies common to the economic literature. Let the alternative space, X, be toplogized as a compact metric space with each Ω_i Range(F) consisting of relations on X whose graphs are closed in X × X. Then the Hausdorff metric toplogy satisfies T1 and T2. More generally, if X is a locally compact, separable metric space and we restict to relations with closed graphs, the topology of closed convergency (Hildenbrand [4], Debreu [2]), also satisfies T1 and T2. Depite the availability of these "natural" topologies, we carry out our development with τ unspecified to maximize generality and to lay bare the roles played by T1 and T2.

III. MAIN RESULTS

Given a topology τ on $\mathcal{B}$, we now state the continuity condition that is central to our results. For x, y ε X,

> $F : \Omega \to \mathcal{B}$ is <u>continuous at (x,y)</u> if $\forall\, \pi \in \Omega$ with $B = F(\pi)$ satisfying xPy, $\exists$ a neighborhood U of π s.t. $xP'y\ \forall \pi' \in U$.

Thus, continuity at (x,y) says that profiles yielding a social preference for x over y must have neighborhoods which preserve this social preference. As a consequence, sufficiently small errors in determining or reporting individual preferences will not alter a social preference for x over y.

<u>Lemma 1</u>. Given $F : \Omega \to \mathcal{B}$, if T2 holds then F continuous $\Rightarrow$ F is continuous at (x,y) $\forall$x, y ε X.

<u>Proof</u>. Given x, y ε X and $B = F(\pi)$ with xPy, T2 gives a neighborhood V of B with $xP'y\ \forall B' \in V$. By continuity of F at π, $\exists$ a neighborhood U of π such that $F(U) \subseteq V$. It then follows that $xP'y\ \forall \pi' \in U$, so F is continuous at (x,y).

Lemma 1 shows that if the topology satisfies T1, standard definition of continuity is stronger than continuity at (x,y). It should be noted that by using continuity at (x,y) rather than continuity, we can henceforward dispense with any further mention of the topological structure

of Range(F), so that T1 and T2 are to be seen as restrictions on the topology for Ω. Our first two theorems will show that this weaker form of continuity is essentially equivalent to the following social choice condition.

> $F : \Omega \to \mathcal{B}$ satisfies <u>invariance with respect to shrinking indifference</u> (ISI) at (x,y) if $\forall \pi, \pi' \in \Omega$,
>
> $xP_iy \Rightarrow xP_i'y$ and $yP_ix \Rightarrow yP_i'x\ \ \forall i \in N$ and xPy implies xP'y.

Thus ISI has an aspect of monotonicity (preserving xPy when individuals shift from indifference to xP_iy) and an aspect of antimonotonicity (preserving xPy when individuals shift from indifference to yP_ix).

<u>Theorem 1</u>. Given T1 and BI at (x,y), then $F : \Omega \to \mathcal{B}$ continuous at (x,y) $\Rightarrow$ F is ISI at (x,y).

<u>Proof</u>. Suppose $xP_iy \Rightarrow xP_i'y$ and $yP_ix \Rightarrow yP_i'x\ \forall i \in N$ and xPy. We will show that xP'y. By continuity at (x,y) $\exists$ a neighborhood U of $\pi = (R_1,R_2,\ldots,R_n)$ such that $xP''y\ \forall\pi'' \in U$. From the way the product topology is defined we can choose neighborhoods U_i of R_i $(i = 1,2,\ldots,n)$ such that $U_1 \times U_2 \times\cdots\times U_n \subseteq U$. Define a profile $\pi^* = (R_1^*,R_2^*,\ldots,R_n^*)$ by using T1 as follows: for i such that $xP_i'y$ but $\sim xP_iy$, choose $R_i^* \in U_i$ such that xP_i^*y. For i such that $yP_i'x$ but $\sim yP_ix$, choose $R_i^* \in U_i$ such that yP_i^*x. For all other i, choose $R_i^* = R_i$. Now $\pi^* \in U \Rightarrow xP^*y$ by the way U was defined. Finally, binary independence then gives xP'y since P' and P* agree on {x,y}.

Theorem 1 has two interesting corollaries. To state the first, we define $F : \Omega \to \mathcal{B}$ to be <u>α - relative majority</u> rule ($1/2 \le \alpha < 1$) if

$$xPy \Leftrightarrow |\{i|xP_iy\}| > \alpha\,(|\{i|xP_iy\}| + |\{i|yP_ix\}|).$$

These rules ignore individual indifference, taking into account only the percentage of concerned voters favoring x over y. The familiar simple majority rule occurs when $\alpha = 1/2$.

Corollary 1. Given T1, an α - relative majority rule F cannot be continuous at any (x,y) with $x \neq y$. If T2 also holds F cannot be continuous.

Proof. First note that α - relative majority rules satisfy BI. Given $x \neq y$, choose $\pi \in \Omega$ such that xP_1y and xI_iy $\forall i > 1$. We must then have xPy. Now choose $\pi' \in \Omega$ such that $xP_1'y$ and $yP_i'x$ $\forall i > 1$. Since $|\{i|xP_i'y\}|/n \leq 1/2$ (we assume $n > 1$), we cannot have $xP'y$, so F does not satisfy ISI at (x,y). By Theorem 1, F cannot be continuous at (x,y). With T2 present, Lemma 1 shows that F cannot, therefore, be continuous.

The second corollary is a continuity based impossibility theorem in the spirit of Arrow. A similar result was developed in McManus [5]. We say that F is strong Pareto (SP) at (x,y) if $\forall\pi \in \Omega$, $\{|i|xP_iy\} \neq \phi$ and $\{i|yP_ix\} = \phi \Rightarrow xPy$.

Corollary 2. Given T1 and any $x \neq y$, there does not exist an $F : \Omega \rightarrow \mathcal{B}$ which satisfies BI, SP, and continuity at (x,y) and (y,x).

Proof. Suppose F satisfied all the above conditions for some $(x,y) \in X \times X$, $x \neq y$. Choose $\pi \in \Omega$ such xP_1y and yI_ix $\forall i > 1$. By SP we have xPy. Now choose $\pi' \in \Omega$ such that $xP_1'y$, $yP_2'x$, and $xI_i'y$ $\forall i > 2$. Since the hypotheses of Theorem 1 hold, F is ISI at (x,y), so we have $xP'y$. A repeat of this argument with the roles of x and y and

individuals 1 and 2 reversed gives a $\pi'' \in \Omega$ with $xP_1''y$, $yP_2''x$, and $xI_i''y$ $\forall i > 2$ and $yP''x$. But BI at (x,y) requires $yP'x$, contradicting the asymmetry of P'.

The following converse to Theorem 1 shows that ISI and continuity at (x,y) are essentially equivalent.

Theorem 2. Given T2, $F : \Omega \rightarrow \mathcal{B}$ satisfies ISI at (x,y) $\Rightarrow$ F is continuous at (x,y).

Proof. Given $\pi \in \Omega$ with xPy, we seek a neighborhood U of π such that $xP'y$ $\forall\pi' \in U$. Let $A = \{i|xP_iy\}$ and $B = \{i|yP_ix\}$. For $i \in A$, T2 ensures a neighborhood U_i of P_i such that $xP_i'y$ $\forall P_i' \in U_i$. For $i \in B$, T2 ensures a neighborhood U_i of P_i such that $yP_i'x$ $\forall P_i' \in U_i$. For $i \in N-(A \cup B)$ let $U_i = \Omega_i$. Then $U = U_1 \times U_2 \times \cdots \times U_n$ is a neighborhood of π. Finally, from ISI at (x,y) it follows that $xP'y$ $\forall\pi' \in U$.

In contrast to Corollary 1, the important class of majority rule procedures do satisfy continuity at each (x,y). Define $F : \Omega \rightarrow \mathcal{B}$ to be α - absolute majority rule ($1/2 \leq \alpha < 1$) if

$$xPy \Leftrightarrow |\{i|xP_iy\}| > \alpha\, n.$$

The ubiquitous absolute majority rule occurs when $\alpha = 1/2$.

Corollary 3. Given T2, the α - absolute majority rules are continuous at every (x,y).

Proof. Just note that these rules satisfy ISI at every (x,y) and apply Theorem 2.

In fact, Theorem 2 implies continity for a much broader class of voting rules, those that have a simple game structure at each (x,y) with $x \neq y$. A (proper, monotonic) simple game at (x,y) over a set N of players is defined by a collection $S(x,y)$ subsets of N satisfying:

S1: $N \in S(x,y)$

S2: $A \in S(x,y)$ and $A \subseteq B \Rightarrow B \in S(x,y)$

S3: $A \in S(x,y) \Rightarrow N - A \notin S(x,y)$.

We say that $F : \Omega \to \mathcal{B}$ is simple at (x,y) if $\exists$ a simple game $S(x,y)$ such that

$$xPy \iff \{i|xP_iy\} \in S(x,y).$$

F is simple if it is simple at (x,y) for all (x,y) and $S(x,y) \equiv S$.

Define $F : \Omega \to \mathcal{B}$ to be semimonotonic at (x,y) if $\forall\pi, \pi' \in \Omega$,

$$\{i|xP_iy\} = \{i|xP_i'y\},\ \{i|xI_iy\} \subseteq \{i|xI_i'y\},\ \text{ and } xPy \Rightarrow xP'y.$$

Finally, call $F : \Omega \to \mathcal{B}$ nontrivial if xPy for some $x, y \in X$ and $\pi \in \Omega$. Theorems 1 and 2 provide the following characterization.

Theorem 3. Given T1 and T2 and $x \neq y$, then a nontrivial $F : \Omega \to \mathcal{B}$ is BI, continuous and semimonotonic at (x,y) $\iff$ F is simple at (x,y).

Proof. The result follows directly from Theorems 1 and 2 as soon as it is observed that F is ISI and semimonotonic at (x,y) $\iff$ F is simple at (x,y).

Thus continuity at (x,y), a seemingly reasonable requirement in some situations, is closely connected with the common notion of a simple game at (x,y). The fact remains, however, that the simple game may change at each pair (x,y), leaving considerable room for variety and complexity.

By combining our results with those presented in Ferejohn, Grether, Matthews, and Packel [3], some local results can be achieved. Given $F : \Omega \to \mathcal{B}$, define F to be continuous-valued at (x,y) if $\forall\pi \in \Omega$, $xPy \Rightarrow \exists$ a neighborhood, W, of (x,y) such that uPv $\forall(u,v) \in W$. This definition requires a topology on X which we assume to be Hausdorff and to have the property that all nonempty open sets contain infinitely many elements. We shall need two additional assumptions on the preference domains Ω_i, $i = 1,2,\ldots,n$.

D1: Given $x \neq y$, $\exists$ neighborhood Y of (x,y) and R, R', R" $\in \Omega_i$ such that uPv, uI'v, and vP"u $\forall(u,v) \in Y$.

D2: Given $x \neq y$ and a sequence $(x_n,y_n) \to (x,y)$ with $(x_n,y_n) \neq (x,y)$ $\forall n$, $\exists$ $R \in \Omega_i$ such that xIy and x_nPy_n $\forall n$.

Finally, we define $F : \Omega \to \mathcal{B}$ to be locally ISI at (x,y) if $\forall\pi, \pi' \in \Omega$, $\exists$ neighborhood W of (x,y) such that $\forall(u,v) \in W$,

$$xP_iy \Rightarrow uP_i'v \text{ and } yP_ix \Rightarrow uP_i'v\ \forall i \in N \text{ and } xPy \text{ implies } uP'v.$$

Theorem 4. Given T1, D1, D2 and BI at (x,y), then $F : \Omega \to \mathcal{B}$ continuous and continuous-valued at (x,y) $\Rightarrow$ F is locally ISI at (x,y).

Proof. Throughout this proof, given $\pi \in \Omega$ let $A_{x,y} = \{i|xP_iy\}$ and $B_{x,y} = \{i|yP_ix\}$ with analogous notation for $\pi' \in \Omega$ and $u, v \in X$. Suppose F were not locally ISI at (x,y). Then $\exists$ π, $\{\pi^n\} \in \Omega$ and $\{x_h\}$, $\{y_h\} \in X$ with $(x_h,y_h) \to (x,y)$ such that xPy, $A^h_{x^h,y^h} \supseteq B^h_{x^h,y^h} \supseteq B_{x,y}$ and $y_hR^hx_h$ $\forall h = 1,2,\ldots$. Since N is a finite set, we can assume with no loss of generality that $A^h_{x^h,y^h} = A'$ and $B^h_{x^h,y^h} = B'$ $\forall h$. We also set

$A_{x,y} = A$ and $B_{x,y} = B$. Using D1 and D2 (and replacing (x_h, y_h) by a subsequence if necessary) $\exists \pi^* \varepsilon \Omega$ such that

i) xP_i^*y and $x_h P_i^* y_h$ $\forall i \varepsilon A$, $\forall h$

ii) yP_i^*x and $y_h P_i^* x_h$ $\forall i \varepsilon B$, $\forall h$

iii) xI_i^*y and $x_h P_i^* y_h$ $\forall i \varepsilon A' - A$, $\forall h$

iv) xI_i^*y and $y_h P_i^* x_h$ $\forall i \varepsilon B' - B$, $\forall h$

v) xI_i^*y $I_i^*x_h$ $I_i^*y_h$ $\forall i \varepsilon N - (A' \cup B')$, $\forall h$.

Since F satisfies BI everywhere,

$$y_h R^h x_h \Rightarrow y_h R^* x_h \quad (R^h \text{ and } R^* \text{ agree on } \{x_h, y_h\}).$$

From continuous-valued at (x,y) we then get yR^*x. Again applying BI (this time at (x,y)), it follows that yRx since R^* and R agree on $\{x,y\}$. This contradicts xPy, establishes the desired contradiction, and completes the proof.

The approach used here is similar in spirit to that of [3]. It is interesting to note that continuity gives ISI at (x,y), while the continuous-valued assumption extends the ISI property on a deleted neighborhood of (x,y). Thus the two continuity properties complement each other nicely. As with our earlier continuity results, we also get a converse theorem.

Theorem 5. Given T2 and continuity of individual preferences in Ω_i $\forall i \varepsilon N$, then $F : \Omega \to \mathcal{B}$ locally ISI at $(x,y) \Rightarrow F$ is continuous and continuous-valued at (x,y).

Proof. Let W be a neighborhood of (x,y) making F locally ISI at (x,y). Given $\pi \varepsilon \Omega$ such that xPy, define $A = \{i | xP_i y\}$ and $B = \{i | yP_i x\}$. Note that each set $P_i \subseteq X \times X$ is open and each $R_i \subseteq X \times X$ is closed since individual preferences are assumed to be continuous. Define

$$Y = W \cap (\bigcap_{i \varepsilon A} P_i) \cap (\bigcap_{i \varepsilon B} (X \times X - R_i)).$$

Then Y is a neighborhood of (x,y) and, by the local ISI assumption at (x,y), $(u,v) \varepsilon Y \Rightarrow uPv$. Thus F is continuous-valued at (x,y). Continuity at (x,y) is taken care of by Theorem 2.

It might be hoped that the combination of F being continuous and continuous-valued at every (x,y) would be enough to imply, under appropriate connectivity assumptions, that F could be completely represented by a single simple game. This does not appear to be true. A global simplicity result requiring an additional assumption is obtained in [3].

IV. DISCUSSION

The results given here suggest that continuous SDP's exist which also satisfy a variety of attractive properties such as monotonicity, anonymity, neutrality, and the weak pareto principle. Indeed, absolute majority rule satisfies each of these conditions and is continuous in a natural sense. Thus, while a number of familiar aggregation procedures such as any of the α-plurality rules, are not continuous in this sense, the imposition of a continuity requirement by itself does not have the effect of precluding "democratic" decision procedures.

These results might seem to be at variance with the work of Chichilnisky, who has argued that a continuous social aggregation rule (SAR) cannot satisfy anonymity and the weak pareto principle [1]. As might be expected, her framework is somewhat different from ours in that preference orderings are represented by functions indicating the most preferred direction at each point in the alternative space. A SAR then associates each n-tuple of preferences with another preference (which may be the zero vector). Her requirement that social preference be representable by indicating a most preferred direction in effect excludes most voting procedures from the start. Such procedures will not generally exhibit a unique socially most preferred direction.

We should also emphasize that, in our framework, there is no contradiction between the continuity property and Arrow's independence of irrelevant alternatives. As corollary 2 makes clear, the continuity of an SDP is incompatible with the strong pareto condition in the presence of binary independence; but this is because the strong pareto principle, by itself, requires the violation of ISI.

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